GAUSSIAN PROCESSES
EXERCISE SHEET 2: LINDEBERG EXCHANGE PRINCIPLE AND HEAVY-TAILED CLT

Exercise 1.

For any $a < b$ and $\epsilon > 0$, where a, b are continuity points of the CDF function of X, we aim to construct a smooth function f_ϵ with bounded first three derivatives such that:

- $\operatorname{supp}(f_\epsilon) \subset (a-\epsilon,\, b+\epsilon)$,
- $f_\epsilon(x) = 1$ for all $x \in [a, b]$
- $\big|\mathbb{P}(a \le X \le b) - \mathbb{E}[f_\epsilon(X)]\big| < \epsilon$.

This is possible because

$$\mathbb{P}\big(X \in (a-\delta, a) \cup (b, b+\delta)\big) \xrightarrow[\delta \to 0]{} 0.$$

Hence, we choose $\delta < \epsilon$ small enough so that

$$\mathbb{P}\big(X \in (a-\delta, a) \cup (b, b+\delta)\big) < \epsilon.$$

Finally, we use a standard mollifier to construct $f_\epsilon \in C^\infty$ supported on $(a-\delta, b+\delta)$ with $f_\epsilon(x) = 1$ for all $x \in [a, b]$.

Now note that $\mathbb{E}[f_\epsilon(X_n)] \ge \mathbb{P}(a \le X_n \le b)$, we have that

$$\limsup_n \mathbb{P}(a \le X_n \le b) \le \limsup_n \mathbb{E}[f_\epsilon(X_n)] = \mathbb{E}[f_\epsilon(X)] \le \mathbb{P}(a \le X \le b) + \epsilon \tag{0.1}$$

The other direction is similar. Thus take ϵ small enough, we have that $\lim_n \mathbb{P}(a \le X_n \le b) = \mathbb{P}(a \le X \le b)$, which means that $X_n \xrightarrow{d} X$.

□

Exercise 2.

Assume that the first derivative of f is bounded by D. For each $k \in \mathbb{N}$, define

$$Z_k := X_k \mathbf{1}_{\{|X_k| > \delta n^{1/2}\}}, \qquad \text{so that } X_k = Y_k + Z_k.$$

For any $x \in \mathbb{R}$ and $k = 1, 2, \ldots, n$, we have

$$\big|f\big(x + n^{-1/2} Z_k\big) - f(x)\big| = \big|n^{-1/2} Z_k f'(\theta)\big| \le D n^{-1/2} |Z_k|, \tag{0.2}$$

for some θ between x and $x + n^{-1/2} Z_k$.

Now take

$$x = n^{-1/2}\Big(Y_1 + Y_2 + \cdots + Y_n + \sum_{i=1}^{k} Z_i\Big), \qquad k = 1, 2, \ldots, n,$$

and sum (0.2) over all k. This yields

$$\big|f\big(n^{-1/2}(X_1 + \cdots + X_n)\big) - f\big(n^{-1/2}(Y_1 + \cdots + Y_n)\big)\big| \le D n^{-1/2} \sum_{k=1}^{n} |Z_k|.$$

Thus, we obtain

$$\Big|\mathbb{E}\big[f\big(n^{-1/2}(X_1+\cdots+X_n)\big)-f\big(n^{-1/2}(Y_1+\cdots+Y_n)\big)\big]\Big| \leq Dn^{-1/2}\sum_{k=1}^{n}\mathbb{E}|Z_k|. \tag{0.3}$$

Note that when $n\to\infty$, $Z_k^2\to 0$ decreasingly, thus we can choose n sufficiently large such that $\mathbb{E}Z_k^2<\delta^2$. Thus $\mathbb{E}|Z_k|\leq \frac{1}{\delta n^{1/2}}\mathbb{E}|Z_k|^2=\delta n^{-\frac{1}{2}}$. Substituting this bound into (0.3) completes the proof. ☐

Exercise 3.

(1) We begin by showing the following lemma.

Lemma 1. *Let* $\varphi(t)=\mathbb{E}e^{itX}$ *be the characteristic function of* X. *Then there exists* $C>0$ *such that*

$$\varphi(t)=1-C|t|^\alpha+O(t^2)$$

as $t\to 0$.

Proof. It is enough to consider $t>0$. Let $G(x)=\mathbb{P}(X>x)=1-\mathbb{P}(X\leq x)$. Since the law of X is symmetric, we have

$$\varphi(t)=\mathbb{E}\cos(tX)=2\mathbb{E}\cos(tX)\mathbf{1}_{\{X\geq 0\}}=-2\int_0^\infty \cos(tx)dG(x).$$

By integration by parts

$$\varphi(t)=-2\big(-\frac{1}{2}+\int_0^\infty t\sin(tx)G(x)\,dx\big)=1-2t\int_0^\infty \sin(tx)G(x)\,dx.$$

Finally note that

$$\begin{aligned}2t\int_0^\infty \sin(tx)G(x)\,dx &= 2t\int_0^1 \sin(tx)G(x)\,dx+t\int_1^\infty \sin(tx)x^{-\alpha}\,dx\\ &= O(t^2)+t^\alpha\int_t^\infty \sin(x)x^{-\alpha}\,dx.\\ &= O(t^2)+t^\alpha\int_0^\infty \sin(x)x^{-\alpha}\,dx-t^\alpha\int_0^t \sin(x)x^{-\alpha}\,dx\\ &= O(t^2)+t^\alpha\int_0^\infty \sin(x)x^{-\alpha}\,dx-t^\alpha O(t^{2-\alpha}).\end{aligned}$$

Thus the claim holds with $C=\int_0^\infty \sin(x)x^{-\alpha}\,dx$. To see that C is actually positive one may write

$$C=\sum_{k=0}^\infty\int_{k\pi}^{(k+1)\pi}\sin(x)x^{-\alpha}\,dx.$$

Note that the sum is alternating and the $2k$th term is larger in absolute value than $2k+1$th term since

$$\begin{aligned}\int_{2k\pi}^{(2k+1)\pi} \sin(x)x^{-\alpha}\,dx &= \int_{(2k+1)\pi}^{(2k+2)\pi} \sin(x-\pi)(x-\pi)^{-\alpha}\,dx \\ &= -\int_{(2k+1)\pi}^{(2k+2)\pi} \sin(x)(x-\pi)^{-\alpha}\,dx \\ &\geq -\int_{(2k+1)\pi}^{(2k+2)\pi} \sin(x)x^{-\alpha}\,dx.\end{aligned}$$

□

The characteristic function of $n^{-1/\alpha}\sum_{i=1}^{n} X_i$ is now given by $\varphi(n^{-1/\alpha}t)^n$. Let us consider a fixed $t > 0$. For large enough n we will have $\varphi(n^{-1/\alpha}t) > 0$ by continuity, and hence we can look at $\log(\varphi(n^{-1/\alpha}t)^n)$. By the lemma this will be

$$n\log(1 - Cn^{-1}t^{\alpha} + O(n^{-2/\alpha}t^2)) = -Ct^{\alpha} + O(n^{1-2/\alpha}t^2) + O(n^{-1}t^{2\alpha}).$$

As $1 - 2/\alpha < 0$, it therefore tends to $-Ct^{\alpha}$ as $n \to \infty$. Taking exponentials shows that

$$\varphi(n^{-1/\alpha}t)^n \to e^{-Ct^{\alpha}}$$

as required.

(2) Let $P(X > x) = \frac{1}{2}x^{-\alpha} + H(x)x^{-\alpha}$ for $x > 0$. Note that $H(x) \to 0$ when $x \to \infty$, we can find $M > 0$ such that $|H(x)| < M$. By the previous problem, we simply need to prove that $2t\int_0^{\infty}\sin(tx)H(x)x^{-\alpha}\,dx = o(t^{\alpha})$ when $t \to 0^+$. Note that

$$\begin{aligned}t\int_0^{\infty}\sin(tx)H(x)x^{-\alpha}\,dx &= t^{\alpha}\int_0^{\infty}\sin(tx)H(x)(tx)^{-\alpha}\,d(tx) \\ &= t^{\alpha}\int_0^{\infty}\sin(x)H(\frac{x}{t})x^{-\alpha}\,dx\end{aligned} \tag{0.4}$$

and that $|\sin(x)H(\frac{x}{t})x^{-\alpha}|$ is dominated by $M|\sin(x)|x^{-\alpha}$, which is integrable when $\alpha > 1$. Furthermore, $\sin(x)H(\frac{x}{t})x^{-\alpha} \to 0$ when $t \to 0$ for any x, by dominated convergence theorem we have that $\int_0^{\infty}\sin(x)H(\frac{x}{t})x^{-\alpha}\,dx \to 0$ and $2t\int_0^{\infty}\sin(tx)H(x)x^{-\alpha}\,dx = o(t^{\alpha})$.

□

Exercise 4.

We construct a symmetric random variable X such that

$$\mathbb{P}(|X| = 2^n) = \frac{1}{n(n+1)}, \qquad n \in \mathbb{N}^*.$$

The idea is to make sure that X is heavy-tailed, since

$$\mathbb{P}(|X| > n) \sim \frac{1}{\log n} \quad \text{as } n \to \infty.$$

Suppose now that $X_i \overset{d}{=} X$. We claim that there does not exist a normalizing sequence $\{b_n\}$ that makes the normalized sums converge. Indeed, from class we know that $\frac{b_{2n}}{b_n}$ must converge, which implies that $(b_{2^n})^{1/n}$ also converges. Hence there exists $c > 1$ such that

$$b_{2^n} < c^n, \qquad \forall n \in \mathbb{N}^*.$$

We first prove the following lemma.

Lemma 2. *For any $C > 0$ and $0 < \alpha < 2$, there exists $\delta > 0$ such that for all $0 < t < \delta$,*

$$\varphi_X(t) < 1 - Ct^\alpha,$$

where φ_X is the characteristic function of X.

Proof. We have

$$\varphi_X(t) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \cos(2^n t).$$

Take $n = \lfloor \log_2 t^{-1+\beta} \rfloor$, where $\beta > 0$ is sufficiently small. For such n,

$$\varphi_X(t) < 1 - \frac{1}{n(n+1)}\big(1 - \cos(2^n t)\big) < 1 - \frac{1}{4n(n+1)}(2^n t)^2 \sim 1 - \frac{t^{2\beta}}{4n^2},$$

for t small enough. Choosing $2\beta << \alpha$ gives the desired bound. ☐

Now choose $\alpha > 0$ sufficiently small such that $2c^{-\alpha} > 1$. For $0 < t < \delta$, the lemma implies

$$\varphi_X\big(b_{2^n}^{-1} t\big)^{2^n} < \big(1 - Ct^\alpha b_{2^n}^{-\alpha}\big)^{2^n} < \big(1 - Ct^\alpha c^{-\alpha n}\big)^{2^n}.$$

Since

$$2^n \log\big(1 - Ct^\alpha c^{-\alpha n}\big) \to -\infty \quad \text{as } n \to \infty,$$

we obtain

$$\varphi_Z(t) = \lim_{n\to\infty} \varphi_X\big(b_{2^n}^{-1} t\big)^{2^n} \leq 0,$$

which is impossible since $\varphi_Z(0) = 1$ and characteristic functions are continuous at 0. Thus, no such normalizing sequence $\{b_n\}$ can exist. ☐